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Analytical Solution to Bi-Linear Spring Mass Systems Free Vibration

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Abstract

Dynamic systems are generally categorized as linear and non-linear systems based on the analysis approaches developed. Bi-linear spring mass systems form an intermediate category which possesses characteristics of both linear and non-linear systems. Analytical solutions to bi-linear systems cannot in general be reduced to a single equation covering the entire motion domain as in the case of linear systems. Nevertheless, the step-wise linear solutions can be obtained within each domain and the solutions can be related to one other from displacement and velocity continuity requirement for mechanical systems. Detailed analytical solution to single degree of freedom bi-linear spring mass system free vibration is presented in this paper towards deriving equations for time period as well as logarithmic decrement. Displacement versus time as well as velocity versus time plots is generated to demonstrate bi-linear system behavior against the linear system behavior. The solution developed is validated against numerical results obtained from finite element analysis. The methodology shall be extended for higher degree of systems with increased complexity compared to linear system solutions which are already well developed. To demonstrate this, two degree of freedom bi-linear spring mass system solutions are also presented.

Keywords: *Bi-Linear spring, free vibration, frequency, logarithmic decrement*

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INTRODUCTION

Bi-linear spring characteristic is a common problem encountered in many engineering vibration problems. The governing equations of motion are not linear in the entire domain of motion due to different spring stiff nesses in the tensile and compressive domain. A lot of research publications have already been made worldwide on non-linear spring-mass systems starting with French mathematician Poincare who later came to known as father of chaos theory. The chaotic behavior exhibited in Duffing's equation was simulated by Ueda [1] using digital and analog computers. The simulation demonstrated the dependence of non-linear systems on initial conditions. Ueda found harmonic, sub-harmonic, ultra-sub-harmonic and chaotic response of damping and forcing amplitude for forced Duffing oscillatory systems. Bi-linear springs are in fact special case of the more general piece-wise linear springs. Authors give examples of piece-wise linear dynamic systems like impact oscillators, spring-mass systems having clearance, pre-loaded compliance systems, elastic beams with non-linear boundary conditions etc [2–12]. Analytical solutions to single degree of freedom systems with piece wise linear behavior was perhaps first investigated by Hartog [13, 14]. Forced responses of piece-wise linear oscillators were published by Shaw and Holmes [15] wherein harmonic, sub-harmonic and chaotic responses of single degree of freedom bi-linear asymmetric oscillators were investigated. Schulman [16] studied ear drum forced vibrations by modelling ear drum as a simple bi-linear spring. Closed form techniques for solution to piece wise linear systems were attempted by only few researchers worldwide due to increased complexity for higher degree of freedom systems. Single equation solutions using heavy side unit step function was derived by Chicurel-Uziel [17]. A different approach based on frequency domain [18] using incremental harmonic balance is used to obtain steady state solutions of dynamic non-linear systems by Xu et al. Extension of incremental harmonic method to piece-wise linear systems was done by Lau and Zhang [19].

A detailed solution of single Degree of Freedom (DoF) bi-linear spring-mass system vibration is presented in this paper with the objective of deriving equations for time period of vibration and logarithmic decrement in line with previous researcher's work by finding individual solutions within each linear domain and relating the individual domain solutions through displacement and velocity continuity equations at motion transition. The solution methodology is extended to two degree of freedom systems as well considering the different configuration states for the springs. Extension to multidegree systems is possible with increased complexity and computational efforts. Comparison of non-linear Finite Element (FE) simulation solution with the analytical solution developed is also presented in this paper. The equations developed for time period as well as logarithmic decrement shall be readily used for single degree of freedom bi-linear spring-mass systems without getting into finite element simulations.

SINGLE DOF UN-DAMPED SYSTEM FREE VIBRATION

A bi-linear spring-mass system is shown in Fig. 1. The spring stiffness is different in the tensile and compressive domain but constant throughout the tensile or compressive domain. The damping coefficient, $'C'$ is assumed constant throughout the motion domain to keep the model simple though solution to bi-linear damper can as well be developed on similar lines.

Governing equations of motion of mass, M in the spring tensile/compressive domain are

$$M\frac{d^2u}{dt^2} + C\frac{du}{dx} + K_1u = 0, \text{ For } u > 0 \text{ (Tensile Domain)} \qquad (1)$$

$$M\frac{d^2u}{dt^2} + C\frac{du}{dx} + K_2u = 0, \text{ For } u < 0 \text{ (Compressive Domain)} \qquad (2)$$

Where, u is the mass displacement along X-direction.

The damping coefficient, C becomes zero for un-damped bi-linear spring mass system. Governing equations of motion reduces to

$$M\frac{d^2u}{dt^2} + K_1u = 0, \text{ For } u > 0 \text{ (Tensile Domain)} \qquad (3)$$

$$M\frac{d^2u}{dt^2} + K_2u = 0, \text{ For } u < 0 \text{ (Compressive Domain)} \qquad (4)$$

At the motion transition point $(u = 0)$ from tensile to compressive domain and vice versa, the displacement, u as well as velocity, $v = \frac{du}{dt}$ are continuous. The displacement equation for an un-damped linear spring-mass system of spring stiffness, K and mass M is given by

$$u = A\,sin(\omega t + \emptyset) \qquad (5)$$

Where, angular velocity $\omega = \sqrt{\frac{K}{M}}$ and A and $\emptyset$ are amplitude and phase to be determined from the initial conditions. Once these constants are determined, the motion of single DoF linear spring-mass system is completely defined. The methodology fails for a bi-linear spring mass system in Fig. 1 as the motion cannot be defined for the entire range with a single equation owing to different spring stiff nesses in the tensile and compressive domains. But it is possible to find solutions separately for each of the domains as the governing equations of motion are linear within each domain.

Displacement as well as velocity equations in tensile/compressive domain for a particular cycle $'i'$ is given by

For $u > 0$ (Tensile Domain)

$$u = A_{2i-1}\,sin(\omega_1 t + \emptyset_{2i-1}) \qquad (6)$$

$$v = \frac{du}{dt} = \omega_1 A_{2i-1}\,cos(\omega_1 t + \emptyset_{2i-1}) \qquad (7)$$

For $u < 0$ (Compressive Domain)

$$u = A_{2i}\,sin(\omega_2 t + \emptyset_{2i}) \qquad (8)$$

$$v = \frac{du}{dt} = \omega_2 A_{2i}\,cos(\omega_2 t + \emptyset_{2i}) \qquad (9)$$

Where, $\omega_1 = \sqrt{\frac{K_1}{M}}$, $\omega_2 = \sqrt{\frac{K_2}{M}}$ and $A_{2i-1}, A_{2i}, \emptyset_{2\mathrm{i}-1}, \emptyset_{2\mathrm{i}}$ are amplitudes and phases of tensile/compressive domain pertaining to i^{th} cycle.

The amplitudes and phases in the above equations are to be determined from initial conditions as well as by using displacement and velocity continuity during motion transition from one domain to another.

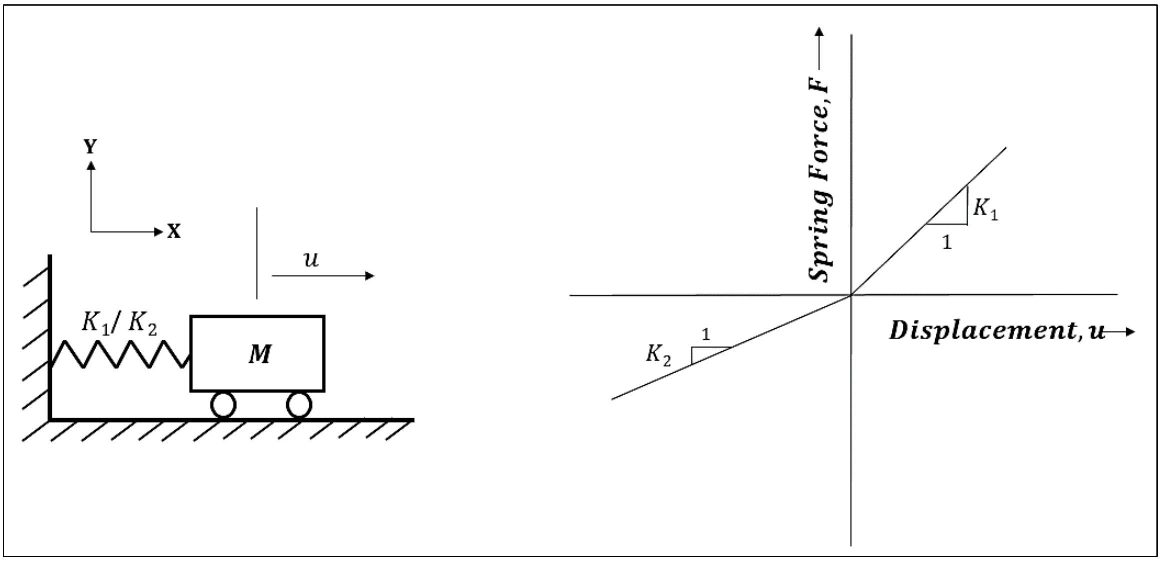

Fig. 1: *Single DoF Bi-linear Spring-Mass System.*

Consider the motion for first cycle ($i = 1$). Let the initial displacement and velocity be $u_0 > 0$ and $v_0 > 0$ at $t = 0$. Choice of motion commencement in the tensile domain is arbitrary.

Equations of motion for the first cycle ($i = 1$) are

For $u > 0$ (Tensile Domain)

$$u = A_1 \, sin(\omega_1 t + \emptyset_1) \quad (10)$$

$$v = \frac{du}{dt} = \omega_1 A_1 \, cos(\omega_1 t + \emptyset_1) \quad (11)$$

For $u < 0$ (Compressive Domain)

$$u = A_2 \, sin(\omega_2 t + \emptyset_2) \quad (12)$$

$$v = \frac{du}{dt} = \omega_2 A_2 \, cos(\omega_2 t + \emptyset_2) \quad (13)$$

Since $u_0 > 0$, motion starts in the tensile region, substituting the initial conditions ($t = 0$) in Eqs. (10) and (11) yield,

$$u_0 = A_1 \, sin \, \emptyset_1 \text{ and } v_0 = \omega_1 A_1 \cos \emptyset_2 \longrightarrow$$

$$A_1 = \sqrt{\left({u_0}^2 + \left(\frac{v_0}{\omega_1}\right)^2\right)} \text{and}$$

$$\emptyset_1 = tan^{-1}\left(\frac{u_0 \omega_1}{v_0}\right)$$

Motion transition from tensile region to compressive region occurs when the displacement becomes zero. The time T_1 when $u = 0$ is obtained from Eq. (10) as

$$0 = A_1 \, sin(\omega_1 T_1 + \emptyset_1) \rightarrow \omega_1 T_1 + \emptyset_1 = \pi$$

Hence

$$T_1 = \frac{\pi - \emptyset_1}{\omega_1} \quad (14)$$

Velocity at motion transition is given by

$$v = \frac{du}{dt} = A_1 \omega_1 \, cos(\omega_1 T_1 + \emptyset_1) = -A_1 \omega_1 \quad (15)$$

This displacement and velocity will be the displacement and velocity for motion in the compressive region due to its continuity.

$$0 = A_2 \, sin(\omega_2 T_1 + \emptyset_2) \quad (16)$$

$$-A_1 \omega_1 = \omega_2 A_2 \, cos(\omega_2 T_1 + \emptyset_2) \quad (17)$$

Eqs. (16) and (17) give $\emptyset_2 = \pi - \omega_2 T_1$ and $A_2 = \frac{A_1 \omega_1}{\omega_2}$

First cycle motion is complete when the displacement u becomes zero again and moves towards $u > 0$. Time, T_2 when motion reversal happens given by,

$$0 = A_2 \, sin(\omega_2 T_2 + \ \emptyset_2) \rightarrow \omega_2 T_2 + \ \emptyset_2 = 2\pi$$

Hence,

$$T_2 = \frac{2\pi - \emptyset_2}{\omega_2} = \frac{\pi + \omega_2 T_1}{\omega_2} \quad (18)$$

Velocity at the beginning of next cycle is given by

$$v = \frac{du}{dt} = A_2 \, \omega_2 cos(\omega_2 T_2 + \ \emptyset_2) = A_2 \, \omega_2 = A_1 \omega_1 \quad (19)$$

Displacement and velocity at beginning of next cycle is then

$$u = 0, v = \frac{du}{dt} = A_1 \omega_1 \quad (20)$$

Proceeding similarly as in the case of cycle 1, it can be shown that, constants A_{2i-1}, A_{2i} for subsequent cycles will be the same as that of cycle 1.

Consider an un-damped single DoF bi-linear spring mass system as defined in Table 1.

Table 1: *Single DoF Un-damped Bi-linear Spring Mass System Definition.*

Item	**Value**	**Units**
M	1.0	Kg
K_1	400.0	N/m
K_2	100.0	N/m

ω_1 and ω_2 are then

$$\omega_1 = \sqrt{\frac{K_1}{M}} = \sqrt{\frac{400}{1}} = 20 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{K_2}{M}} = \sqrt{\frac{100}{1}} = 10 \text{ rad/s}$$

Let the initial displacement and velocity be $u_0 = 1 \times 10^{-2}$ m and $v_0 = 1$ m/s, respectively. Since displacement is positive, the motion starts in the tensile domain. The amplitude and phase pertaining to first and second cycle for tensile domain as well as compressive domain are tabulated in Table 2.

Free vibration displacement-time as well as velocity-time plots for an un-damped single DoF linear system ($K = K_1$ or K_2 for both tensile as well as compressive domains) as well as bi-linear spring mass system are given in Fig. 2 and Fig. 3, respectively. Plots are made non-dimensional by dividing the displacement and velocity with initial displacement and initial velocity, respectively. Also, time is made non-dimensional with respect to time period, T. Both the plots show a different behavior for bi- linear spring mass system free vibration compared to that of linear spring mass systems. The velocity–time plot has got identical amplitudes in both spring mass systems though the time period is different. For bi-linear spring displacement-time plots, the time period as well as amplitude in the compressive domain is different from linear spring displacement-time plots. The displacement amplitude is same for both spring mass systems in the tensile domain. This is because, the initial conditions are defined in the tensile domain and identical conditions would have obtained had the motion started in the compressive domain.

Table 2: *Single DoF Un-damped Amplitude and Phase Angles for First and Second Cycle.*

Item	**Value**	**Units**
Tensile Domain		
A_1	5.1×10^{-2}	m
$\emptyset_1$	0.20	Radians
A_3	5.1×10^{-2}	m
$\emptyset_3$	-2.94	Radians
Compressive Domain		
A_2	1×10^{-1}	m
$\emptyset_2$	1.67	Radians
A_4	1×10^{-1}	m
$\emptyset_4$	3.24	Radians

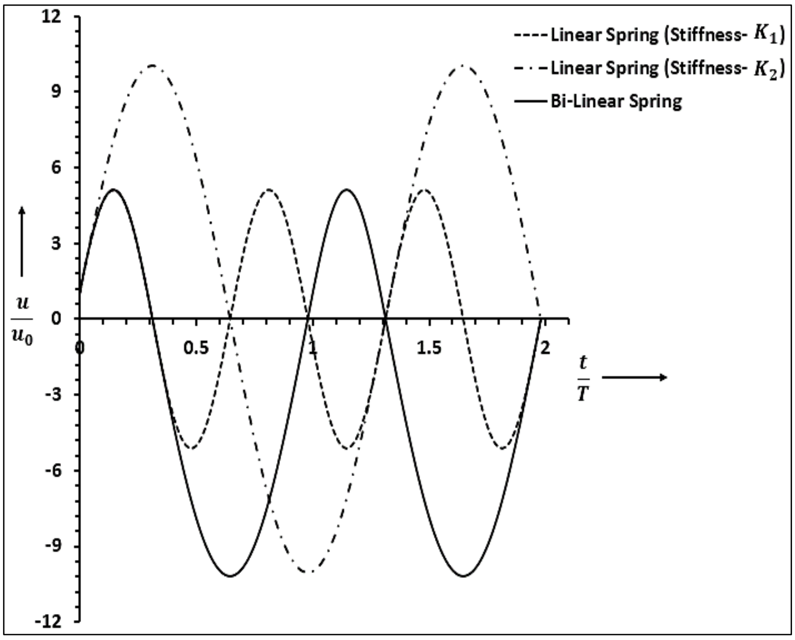

Fig. 2: *Single DoF Un-damped System Free Vibration Displacement-Time Plots.*

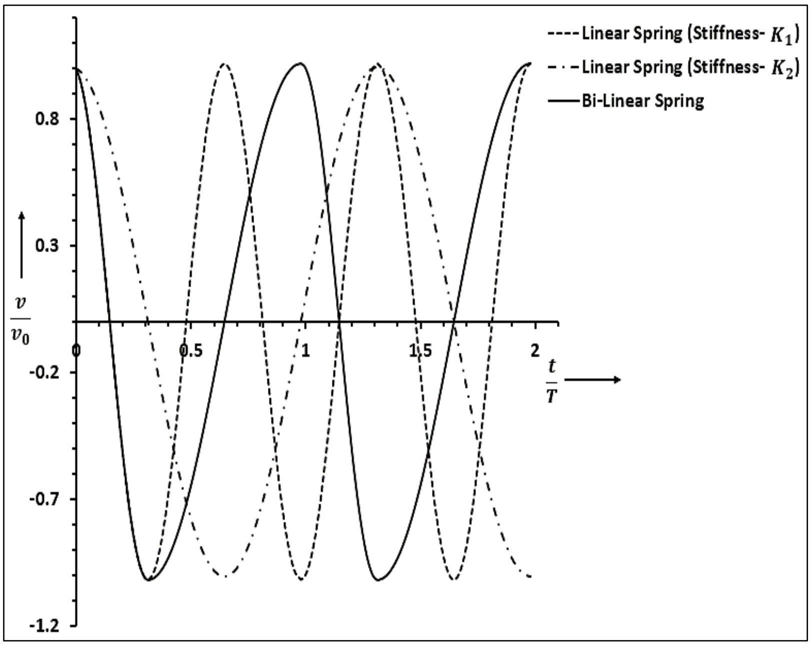

***Fig. 3:** Single DoF Un-damped System Free Vibration Velocity-Time Plots.*

Single DoF Un-Damped Free Vibration Time Period Estimation

Let T be the time period of un-damped free vibration of bi-linear spring mass system. Time required for completion of first cycle is T_2. Time period is given by the summation of time corresponding to phase, $\emptyset_1$ and T_2.

$$T = T_2 + \frac{\emptyset_1}{\omega_1} = \frac{\pi+\omega_2 T_1}{\omega_2} + \frac{\pi}{\omega_1} - T_1 = \frac{\pi}{\omega_1} + \frac{\pi}{\omega_2} \quad (21)$$

Angular velocity, $\omega_{bi-linear}$ of the bi-linear spring mass system is defined as

$$\omega_{bi-linear} = \frac{2\pi}{T} \quad (22)$$

Substituting T from Eq. (21) in to Eq. (22) gives

$$\omega_{bi-linear} = \frac{2\pi}{\frac{\pi}{\omega_1}+\frac{\pi}{\omega_2}} = \frac{2\omega_1\omega_2}{\omega_1+\omega_2} \quad (23)$$

SINGLE DOF DAMPED SYSTEM FREE VIBRATION

Damped systems free vibration motion is governed by the amount of damping present in the system. Only under-damped system in both tensile and compressive domain are considered as motion transition from tensile to compressive domain and vice versa does not happen in over-damped and critically damped systems.

Governing equations of motion of mass, M in the spring tensile/ compressive domain for damped bi-linear spring mass system are given by Eqs. (1) and (2). Damping ratios for tensile as well as compressive domains are defined as

$$\zeta_1 = \frac{C}{2M\omega_1}, \text{ For } u > 0 \text{ (Tensile Domain)} \quad (24)$$

$$\zeta_2 = \frac{C}{2M\omega_2}, \text{ For } u < 0 \text{ (Compressive Domain)} \quad (25)$$

At the motion transition point from tensile to compressive domain and vice versa $(u = 0)$, the displacement, u as well as velocity, $v = \frac{du}{dt}$ are continuous.

A single equation will not define the motion of the damped bi-linear system in the entire motion domain. Solution for each of the domain for a particular cycle 'i' is given by

For $u > 0$

$$u = A_{2i-1} e^{-\zeta_1\omega_1 t} sin\,(\omega_{d1}t + \emptyset_{2i-1}) \quad (26)$$

$$v = \frac{du}{dt} = A_{2i-1}\omega_{d1} e^{-\zeta_1\omega_1 t} cos\,(\omega_{d1}t + \emptyset_{2i-1}) - \zeta_1\omega_1 u \quad (27)$$

For $u < 0$

$$u = A_{2i} e^{-\zeta_2\omega_2 t} sin\,(\omega_{d2}t + \emptyset_{2i}) \quad (28)$$

$$v = \frac{du}{dt} = A_{2i}\omega_{d2} e^{-\zeta_2\omega_2 t} cos\,(\omega_{d2}t + \emptyset_{2i}) - \zeta_2\omega_2 u \quad (29)$$

Where,

$$\omega_{d1} = \omega_1\sqrt{\left(1-{\zeta_1}^2\right)} \text{ and } \omega_{d2} = \omega_2\sqrt{\left(1-{\zeta_2}^2\right)}$$

$A_{2i-1}, A_{2i}, \emptyset_{2i-1}, \emptyset_{2i}$ are constants and phase angles, respectively for tensile and compressive domains corresponding to cycle 'i'.

Consider the motion for first cycle ($i = 1$). Let the initial displacement and velocity be $u_0 > 0$ and $v_0 > 0$ at $t = 0$. Since $u_0 > 0$, motion starts in the tensile region, substituting the initial conditions ($t = 0$) in Eqs. (26) and (27) yield,

$$u_0 = A_1 sin\, \emptyset_1 \quad (30)$$

$$v_0 = A_1\omega_{d1} cos\, \emptyset_1 - \zeta_1\omega_1 u_0 \quad (31)$$

Hence, constant A_1 and phase $\emptyset_1$ for tensile domain in the first cycle are given by

$$A_1 = \sqrt{\left({u_0}^2 + \left(\frac{v_0+\zeta_1\omega_1 u_0}{\omega_{d1}}\right)^2\right)} \text{ and}$$

$$\emptyset_1 = tan^{-1}\left(\frac{u_0\omega_{d1}}{v_0+\zeta_1\omega_1 u_0}\right)$$

Motion transition from tensile region to compressive region occurs when the displacement becomes zero. The time T_1 when $u = 0$ is obtained from Eq. (26) as

$$0 = A_1 e^{-\zeta_1\omega_1 T_1} sin\,(\omega_{d1}T_1 + \emptyset_1)$$

$$\rightarrow sin(\omega_{d1}T_1 + \emptyset_1) = 0$$

$$\rightarrow \omega_{d1}T_1 + \emptyset_1 = \pi$$

Hence,

$$T_1 = \frac{(\pi-\emptyset_1)}{\omega_{d1}} \quad (32)$$

From Eq. (27), velocity at motion reversal is given by

$$v = -A_1\omega_{d1}e^{-\zeta_1\omega_1 T_1} \quad (33)$$

At the onset of motion in compressive domain, the displacement initial condition is given by Eq. (28) as

$$A_2 e^{-\zeta_2\omega_2 T_1} sin\,(\omega_{d2}T_1 + \emptyset_2) = 0$$

Hence,

$$\omega_{d2}T_1 + \emptyset_2 = \pi$$

Substituting T_1 from Eq. (32), phase for compressive domain for first cycle,

$$\emptyset_2 = \pi - \frac{\omega_{d2}}{\omega_{d1}}(\pi - \emptyset_1) \quad (34)$$

Velocity at the onset of motion in compressive domain for first cycle is then (Eq. (29))

$$v = -A_2\omega_{d2}e^{-\zeta_2\omega_2 T_1} = -A_1\omega_{d1}e^{-\zeta_1\omega_1 T_1} \quad (35)$$

It can be shown that, $\zeta_1\omega_1 = \zeta_2\omega_2 = \frac{c}{2m}$.

Therefore A_2 is given by

$$A_2 = A_1\frac{\omega_{d1}}{\omega_{d2}} \quad (36)$$

First cycle motion is complete when the displacement u becomes zero again and moves towards $u > 0$. Let T_2 be the time when motion reversal happens given by Eq. (28) as

$$A_2 e^{-\zeta_2\omega_2 T_2} sin\,(\omega_{d2}T_2 + \emptyset_2) = 0 \rightarrow T_2 = \frac{(2\pi-\emptyset_2)}{\omega_{d2}} = \frac{\pi}{\omega_{d2}} + \frac{(\pi-\emptyset_1)}{\omega_{d1}} \quad (37)$$

Velocity at onset of second cycle motion is then ((Eq. (29)),

$$v = A_2\omega_{d2}e^{-\zeta_2\omega_2 T_2} = A_1\omega_{d1}e^{-\zeta_2\omega_2 T_2} \quad (38)$$

Proceeding similarly as in the case of cycle 1, the amplitudes and phases of subsequent cycles can be evaluated. It can be shown that, constants A_{2i-1}, A_{2i} for subsequent cycles will be the same as that of cycle 1.

Consider a damped bi-linear spring mass system as defined in Table 3.

Table 3: *Single DoF Damped Bi-linear Spring Mass System Definition.*

Item	**Value**	**Units**
M	1.0	Kg
K_1	400.0	N/m
K_2	100.0	N/m
C	5.0	Ns/m

Angular velocities as well as damping ratios are tabulated in Table 4.

Table 4: *Single DoF Damped System Angular Velocities and Damping Ratios.*

Item	**Value**	**Units**
ω_1	20.00	Rad/s
ω_2	10.00	Rad/s
ζ_1	0.13	-
ζ_2	0.25	-
ω_{d1}	19.84	Rad/s
ω_{d2}	9.68	Rad/s

Let the initial displacement and velocity be the same as that of un-damped free vibration in preceding section for un-damped free vibration. Since displacement is positive the motion starts in the tensile domain. The constants A_{2i-1}, A_{2i} and phase angles $\emptyset_{2i-1}, \emptyset_{2i}$ for first and second cycle for tensile domain as well as compressive domain are tabulated in Table 5.

Displacement-time as well as velocity-time plots for damped bi-linear mass spring system free vibration are given in Fig. 4 and Fig. 5, respectively. Linear spring stiff ness (i.e., $k = K_1$ or K_2 for both tensile as well as compressive domains) plots are also given for comparison. Displacement, velocity and time are made non-dimensional with respect to initial displacement, velocity and time period, respectively. As in the case of un-damped systems, different behavior is observed for displacement as well as velocity-time plots for damped bi- linear mass spring systems.

Single Dof Damped Free Vibration Time Period Estimation

Let T_d be the time period. Time required for completion of first cycle is T_2. Time period is given by the summation of time corresponding to phase $\emptyset_1$ and T_2.

$$T_d = T_2 + \frac{\emptyset_1}{\omega_{d1}}$$

From Eq. (37),

$$T_d = \frac{\pi}{\omega_{d2}} + \frac{(\pi-\emptyset_1)}{\omega_{d1}} + \frac{\emptyset_1}{\omega_{d1}} \rightarrow T_d = \frac{\pi}{\omega_{d2}} + \frac{\pi}{\omega_{d1}} \quad (39)$$

Angular velocity, $\omega_{d(bi-linear)}$ of the damped bi-linear spring mass system is defined as

$$\omega_{d(bi-linear)} = \frac{2\pi}{T_d}$$

Substituting $\mathrm{T_d}$ from Eq. (39)

$$\omega_{d(bi-linear)} = \frac{2\pi}{\frac{\pi}{\omega_{d2}}+\frac{\pi}{\omega_{d1}}} = \frac{2\omega_{d1}\omega_{d2}}{\omega_{d1}+\omega_{d2}} \quad (40)$$

Table 5: *Single DoF Damped System Constants and Phase Angles for First and Second Cycle.*

Item	Value	Units
Tensile Domain		
A_1	5.0×10^{-2}	m
$\emptyset_1$	0.19	Radians
A_3	5.0×10^{-2}	m
$\emptyset_3$	-3.11	Radians
Compressive Domain		
A_2	1.1×10^{-1}	m
$\emptyset_2$	1.70	Radians
A_4	1.1×10^{-1}	m
$\emptyset_4$	3.31	Radians

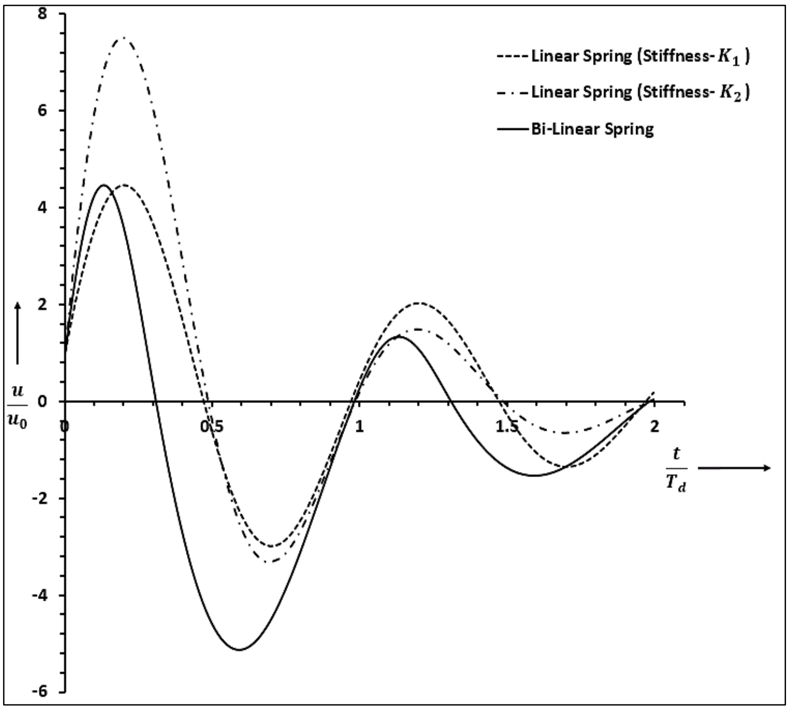

Fig. 4: *Single DoF Damped System Free Vibration Displacement-Time Plots.*

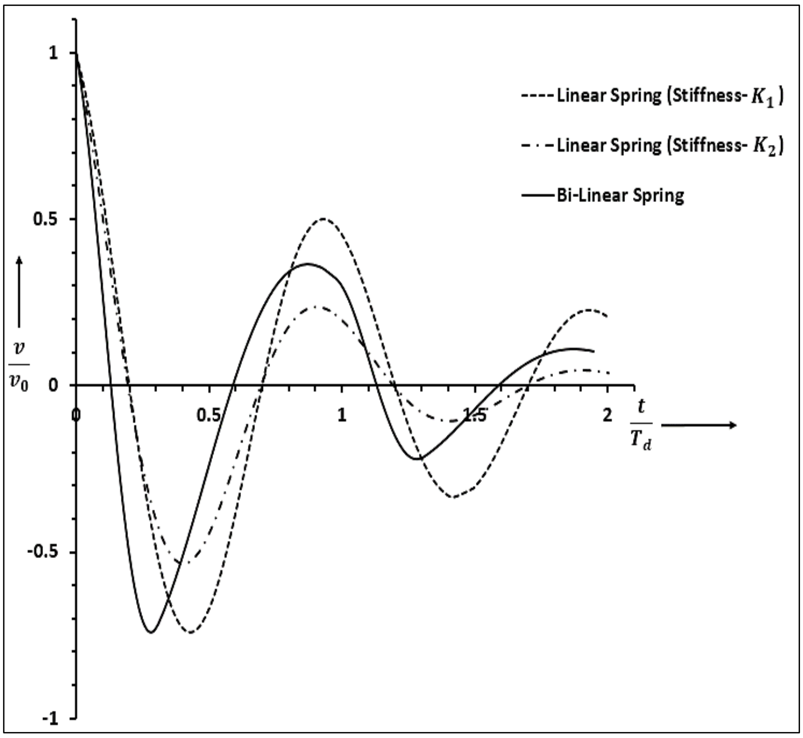

Fig. 5: *Single DoF Damped System Free Vibration Velocity-Time Plots.*

Single Dof Damped Free Vibration Logarithmic Decrement Estimation

Logarithmic decrement is defined as natural logarithm of ratio of maximum amplitude of a cycle to the maximum amplitude of succeeding cycle. Let T_i be the time corresponding to maximum amplitude in the tensile domain for i^{th} cycle. Time T_{i+1} for time corresponding to maximum amplitude in the tensile domain for $(i+1)^{th}$ cycle is given by.

$T_{(i+1)} = T_i + T$, where, $'T'$ is the time period of damped vibration.

Maximum amplitude is when the sine term becomes unity in Eq. (26)

For i^{th} cycle

$$u_{i\_max} = A_1 e^{-\zeta_1 \omega_1 T_i} \tag{41}$$

For $(i+1)^{th}$ cycle

$$u_{(i+1)\_max} = A_1 e^{-\zeta_1 \omega_1 T_{(i+1)}} \tag{42}$$

$$\delta = ln\left(\frac{u_{i\_max}}{u_{(i+1)\_max}}\right) = ln\left(\frac{A_1 e^{-\zeta_1 \omega_1 T_i}}{A_1 e^{-\zeta_1 \omega_1 T_{(i+1)}}}\right) = \zeta_1 \omega_1 T \tag{43}$$

Substituting T from eqn. (39) and using relation $\zeta_1 \omega_1 = \zeta_2 \omega_2$

$$\delta = \frac{\pi \zeta_2 \omega_2}{\omega_{d2}} + \frac{\pi \zeta_1 \omega_1}{\omega_{d1}} \longrightarrow \delta = \frac{\pi \zeta_1}{\sqrt{(1-{\zeta_1}^2)}} + \frac{\pi \zeta_2}{\sqrt{(1-{\zeta_2}^2)}} \tag{44}$$

EXTENSION TO TWO DOF SYSTEMS

The solution methodology for single degree of freedom bi-linear spring mass system is extended for two DoF bi-linear spring systems with increased complexity. A two DoF bi-linear mass spring system is shown in Fig. 6. To reduce complexity linear viscous dampers are assumed along with bi-linear springs-1 and 2.

Four different possible spring configuration states are

a) Spring-1 in tensile domain, Spring-2 in tensile domain
b) Spring-1 in tensile domain, Spring-2 in compressive domain
c) Spring-1 in compressive domain, Spring-2 in tensile domain
d) Spring-1 in compressive domain, Spring-2 in compressive domain

Spring-1 is in tensile domain if displacement $u_1 > 0$ and in compressive domain if $u_1 < 0$. Spring-2 domain is determined by the relative displacement u_2-u_1, in tensile domain if u_2-$u_1 > 0$ and in compressive domain if u_2-$u_1 < 0$.

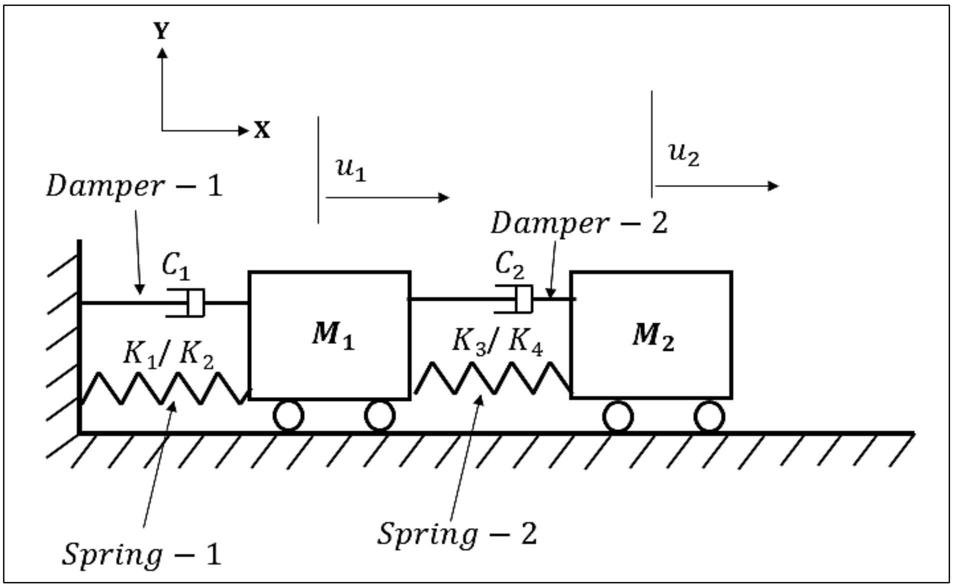

Fig. 6: *Two DoF Bi-linear Spring-Mass System.*

The spring mass system will be linear within each of the configurations listed and standard solutions are available within the domain as a linear combination of Eigen vectors. The standard solution for un-damped ($C_1 = C_2 = 0$) systems are given by

$$u_1^k = R_1^k \, sin(\omega_1^k t + \phi_1^k) + R_2^k \, sin(\omega_2^k t + \phi_2^k) \quad (45)$$

$$v_1^k = \frac{du_1^k}{dt} \quad (46)$$

$$u_2^k = r_i^k R_1^k \, sin(\omega_1^k t + \phi_1^k) + r_2^k R_2^k \, sin(\omega_2^k t + \phi_2^k) \quad (47)$$

$$v_2^k = \frac{du_2^k}{dt} \quad (48)$$

Where, R_1^k, R_2^k, ϕ_1^k, ϕ_2^k, $\omega_1^k, \omega_2^k, r_i^k, r_2^k$ are the amplitudes, phases, Eigen values, Eigen vector ratios (ratio of Eigen vector of M_2 to that of M_1), respectively for k^{th} configuration. As in the case of single degree of freedom system, the amplitudes and phase angles will vary with domain transition and hence need to be calculated from displacement and velocity continuity during domain transition.

For under damped systems, the Eigen values as well as Eigen vector ratios will be complex numbers in general. The standard solutions for damped two degree of freedom systems are given below.

$$u_1^k = e^{\alpha_1^k t}(A^k \cos \beta_1^k t + B^k \sin \beta_1^k t) + e^{\alpha_2^k t}(C^k \cos \beta_2^k t + D^k \sin \beta_2^k t) \quad (49)$$

$$v_1^k = \frac{du_1^k}{dt} \quad (50)$$

$$u_2^k = e^{\alpha_1^k t}\left(\left(\gamma_1^k A^k + \delta_1^k B^k\right) \cos \beta_1^k t + \left(\gamma_1^k B^k - \delta_1^k A^k\right) \sin \beta_1^k t\right) + e^{\alpha_2^k t}\left(\left(\gamma_2^k C^k + \delta_2^k D^k\right) \cos \beta_2^k t + \left(\gamma_2^k D^k - \delta_2^k C^k\right) \sin \beta_2^k t\right) \quad (51)$$

$$v_2^k = \frac{du_2^k}{dt} \quad (52)$$

Where, $\alpha_1^k, \alpha_2^k, \beta_1^k, \beta_2^k$ are the real and imaginary parts of Eigen values for k^{th} configuration. $\gamma_1^k, \gamma_2^k, \delta_1^k, \delta_2^k$ are the real and imaginary parts of the corresponding Eigen vector ratios. A^k, B^k, C^k and D^k are solution coefficients for k^{th} configuration which vary with domain transition and hence need to be calculated from displacement and velocity continuity during domain transition.

Consider a two degree of freedom bi-linear spring mass system as defined in Table 6.

Table 6: *Two DoF Un-damped Spring Mass System Definition.*

Item	**Value**	**Units**
M_1	2.0	Kg
K_1	400.0	N/m
K_2	300.0	N/m
M_2	1.0	Kg
K_3	200.0	N/m
K_4	100.0	N/m

C_1	5.0	Ns/m
C_2	5.0	Ns/m

Eigen values and eigen vector ratios corresponding to four configuration states for un-damped as well as damped systems are tabulated in Table 7 and Table 8, respectively.

Let the initial displacements and velocities be $u_1^0 = 1.5 \times 10^{-2}$ m, $u_2^0 = 2.5 \times 10^{-2}$ m, $v_1^0 = 1$ m/s and $v_2^0 = 2$ m/s, respectively for M_1 and M_2. The choice of initial conditions are arbitrary. The displacement-time as well as velocity –time plots are provided in Fig. 7 to Fig. 10 for both damped and un-damped two degree of freedom systems. The displacements as well as velocities are made non-dimensional with respect to initial displacements as well as initial velocities. Time is made non-dimensional with respect to time of motion, T_{total} considered.

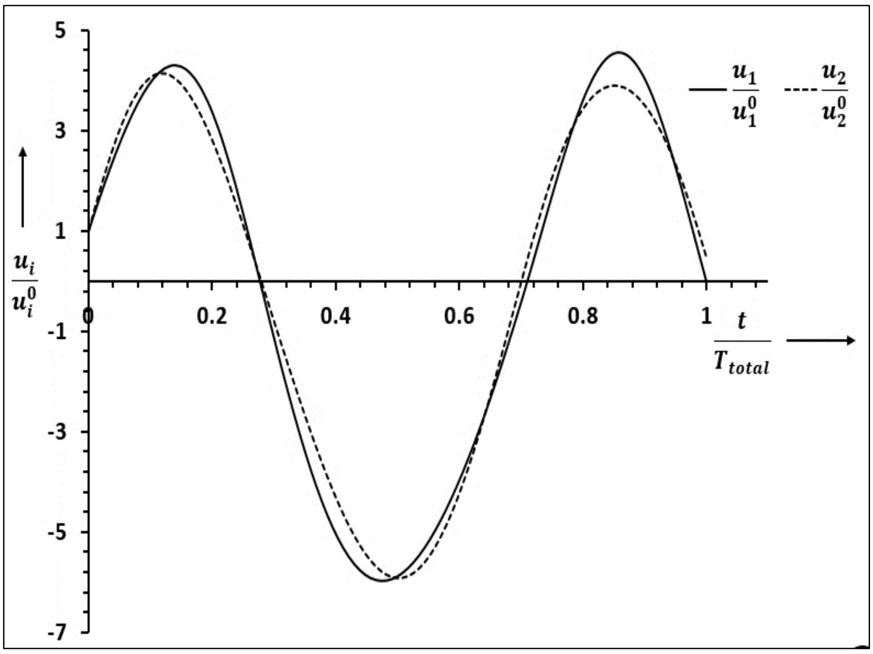

Fig. 7: *Two DoF Un-damped System Free Vibration Displacement-Time Plots.*

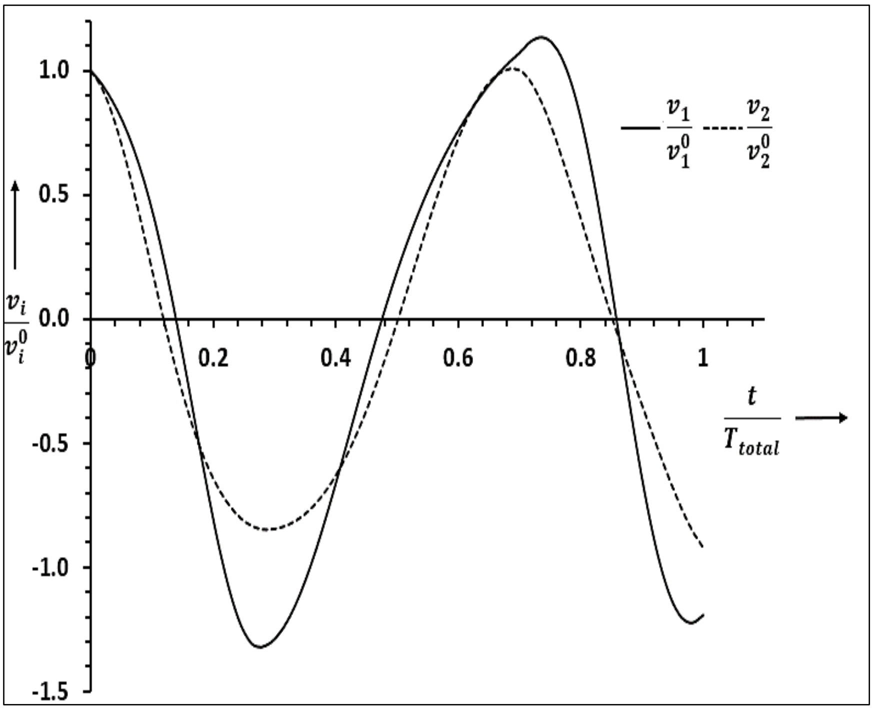

Fig. 8: *Two DoF Un-damped System Free Vibration Velocity -Time Plots.*

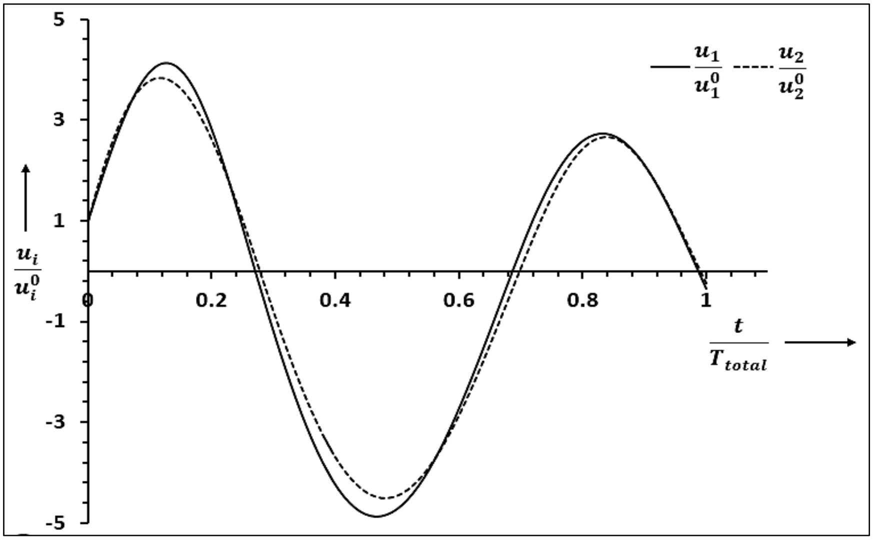

Fig. 9: *Two DoF Damped System Free Vibration Displacement-Time Plots.*

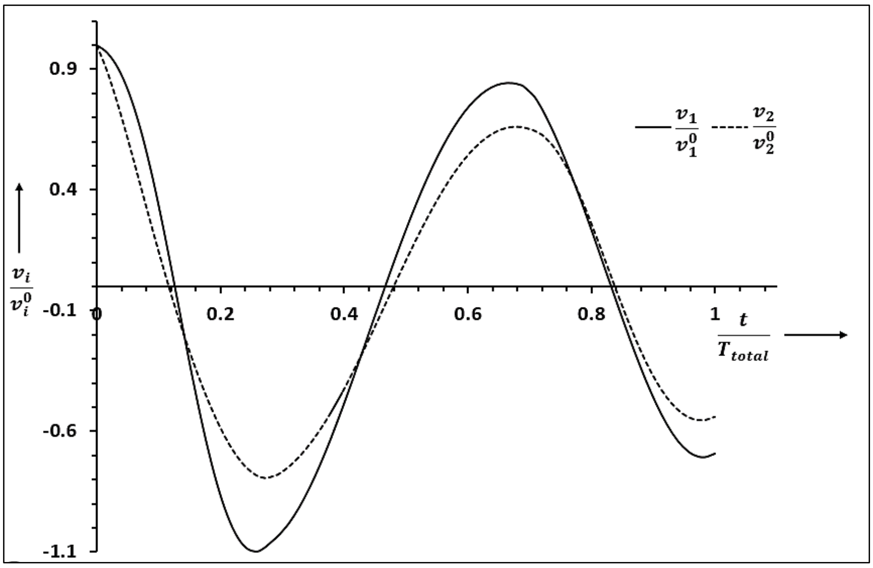

Fig. 10: *Two DoF Damped System Free Vibration Velocity-Time Plots.*

Table 7: *Eigen Values and Eigen Vectors for Un-damped Two DoF System.*

Configuration	ω_1^k	ω_2^k	r_i^k	r_2^k
Spring-1 tensile, Spring-2 tensile ($k = 1$)	10.0	20.0	2.0	-1.0
Spring-1 tensile, Spring-2 compressive ($k = 2$)	8.48	16.68	3.56	-0.56
Spring-1 compressive, Spring-2 tensile ($k = 3$)	11.04	18.11	1.28	-0.78
Spring-1 compressive, Spring-2 compressive ($k = 4$)	7.96	15.38	2.73	-0.73

Table 8: *Eigen Values and Eigen Vectors for Damped Two DoF System.*

Configuration	α_1^k, β_1^k	α_2^k, β_2^k	γ_1^k, δ_1^k	γ_2^k, δ_2^k
Spring-1 tensile, Spring-2 tensile ($k = 1$)	-0.83, 9.99	-4.17, 19.51	1.97, -0.16	-1.03, -0.17
Spring-1 tensile, Spring-2 compressive ($k = 2$)	-1.30, 8.53	-3.7, 15.97	3.22, -1.03	-0.59, -0.36
Spring-1 compressive, Spring-2 tensile ($k = 3$)	-0.18, 10.81	-4.82, 17.86	1.36, 0.13	-1.01, -0.43
Spring-1 compressive, Spring-2 compressive ($k = 4$)	-1.06, 7.99	-3.94, 14.67	2.54, -0.58	-0.79, -0.33

COMPARISON WITH FE SIMULATION RESULTS

Bi-linear free vibration problem solutions given in preceding sections are compared with Finite Element (FE) simulations. Nastran non-linear transient analysis (SOL 129) is used for simulation of un-damped as well as damped bi-linear spring mass systems. The spring mass system parameters as well as initial conditions are same as defined in respective preceding sections for single DoF as well as two DoF systems. The comparison displacement plots are given in Fig. 11 to Fig. 14, respectively. The displacement-time plots show that analytical solutions are closely matching with FE simulation results for both single degree and two degree of freedom bi-linear spring-mass systems.

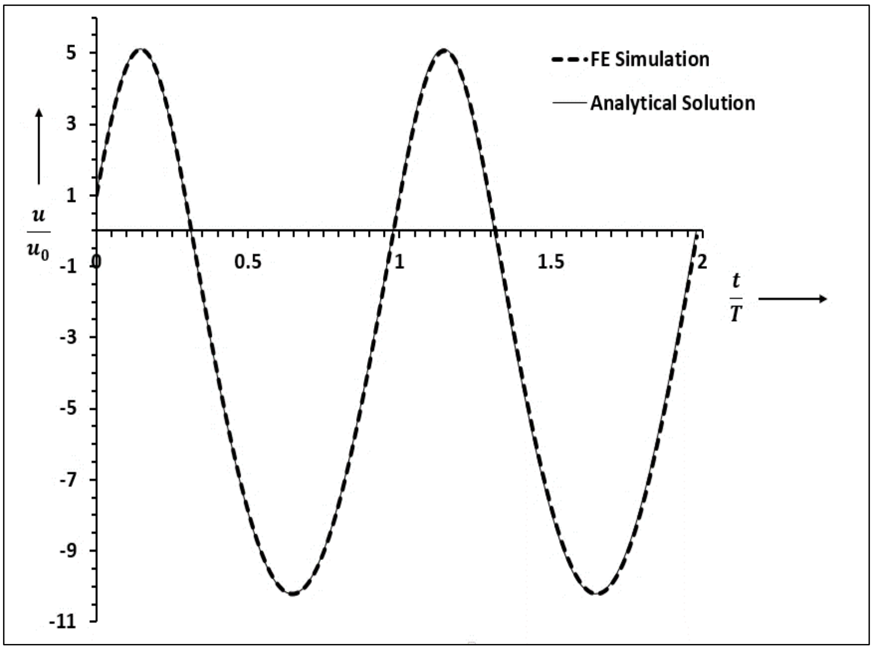

Fig. 11: *Single DoF Un-damped Free Vibration Displacement-Time Comparison Plot.*

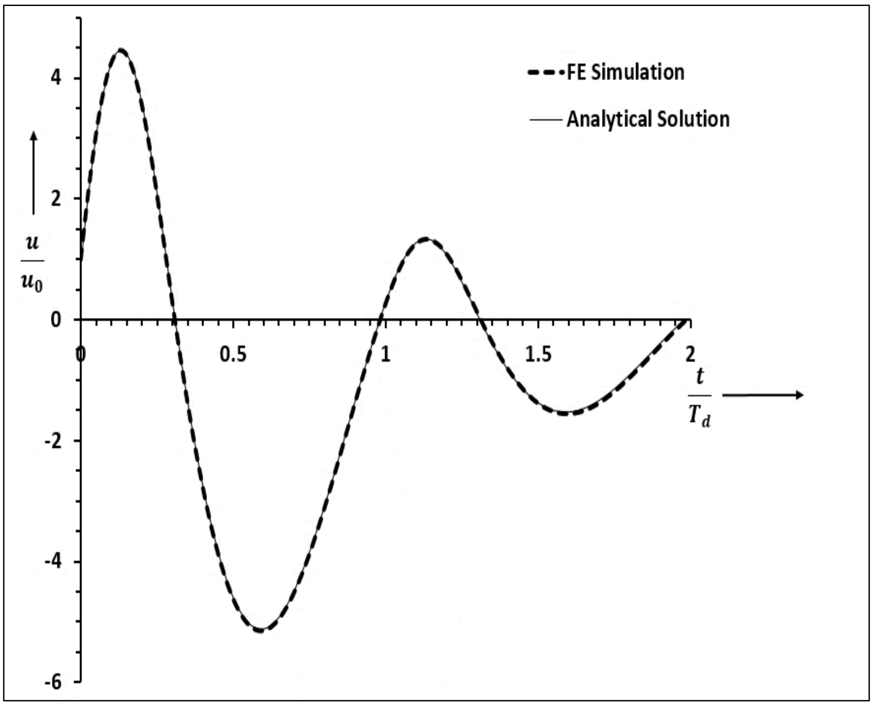

Fig. 12: *Single DoF Damped Free Vibration Displacement-Time Comparison Plot.*

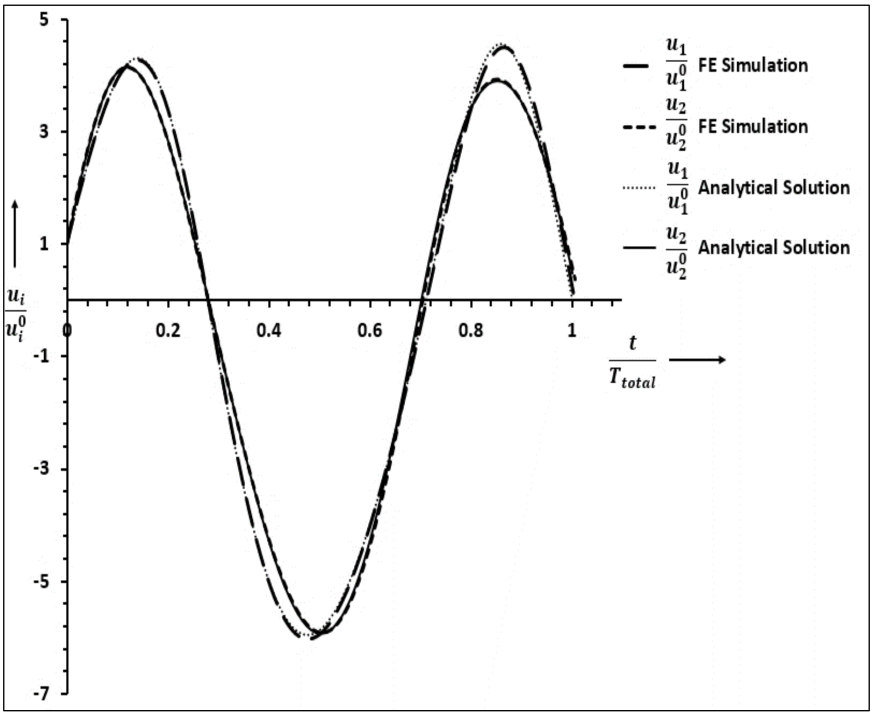

__Fig. 13:__ Two DoF Un-damped Free Vibration Displacement-Time Comparison Plot.

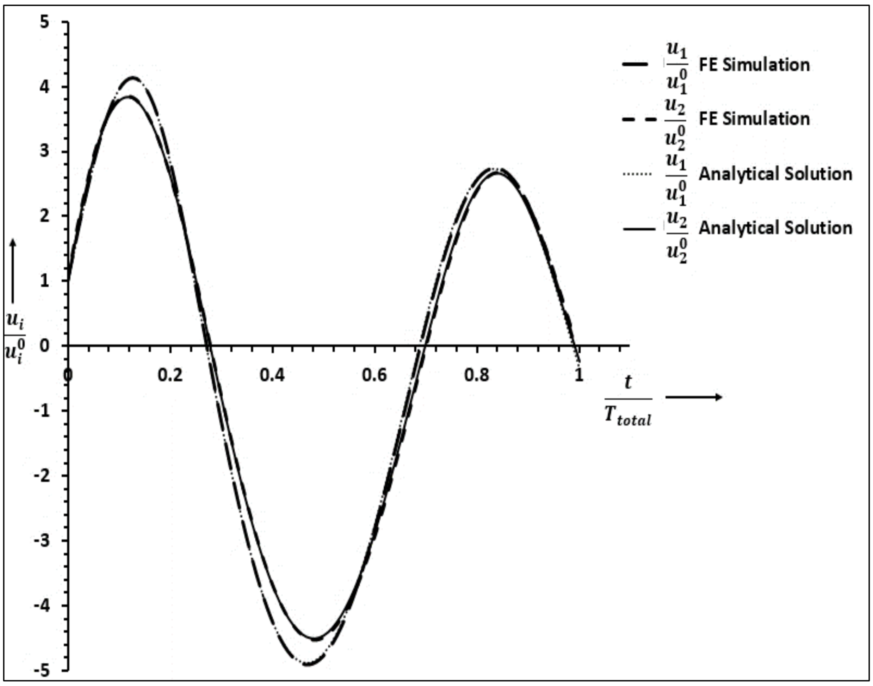

__Fig. 14:__ Two DoF Damped Free Vibration Displacement-Time Comparison Plot.

CONCLUSIONS AND FUTURE SCOPE

Detailed analytical solutions for free vibration of single degree of freedom bi-linear spring-mass systems have been presented in this paper with derivations of time period as well as logarithmic decrement. Extension of the methodology to two degree of freedom bi-linear spring mass systems is also demonstrated. Bi-linear free vibration problem solution methodology can further be extended to forced vibration problems both periodic and non-periodic. For periodic harmonic excitations, conventional solutions for linear systems can be extended for bi-linear spring systems as well. It will be interesting to find the solution and displacement-time plots for un-damped forced vibration when the exciting frequency matches with one of the domain (tensile/compressive) frequencies. In such a case there will be resonance condition in one domain and non-resonance condition for the other. Non-harmonic periodic excitations can be transformed to harmonic excitations through Fourier transformations. In the case of non-periodic excitations, the convolution integral method applicable for linear systems can no longer be used for bi-linear springs as the bi-linear spring systems are no longer linear in the entire domain of motion. Nevertheless, the solution can still be made through conventional particular integral solution for standard functions of excitation. The solution methodology can also be extended to higher degree of freedom systems with increased complexity.

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NOMENCLATURE

DoF	:	Degree of Freedom
FE	:	Finite Element
M	:	Mass
M_i	:	2 DoF Mass, $i = 1,2$
K_i	:	Spring Stiffness, $i = 1,2,3,4$
C	:	Damping Coefficient
C_i	:	2 DoF Damping Coefficient, $i = 1,2$
u	:	Displacement
$\mathrm{u_i}$	:	Displacement of 2 DoF system masses, $i = 1,2$
v	:	Velocity
v_i	:	Velocity of 2 DoF system masses, $i = 1,2$
t	:	Time
u_0	:	Initial Displacement
v_0	:	Initial Velocity
u_i^0	:	2 DoF Free Vibration Initial Displacement, $i = 1,2$
v_i^0	:	2 DoF Free Vibration Initial Displacement, $i = 1,2$
ω_i	:	Un-damped Free Vibration Angular Velocity, $i = 1,2$
ω_{di}	:	Damped Free Vibration Angular Velocity, $i = 1,2$
$\omega_{bi-linear}$	:	Bi-linear Spring Un-damped Angular Velocity
$\omega_{d(bi-linear)}$	:	Bi-linear Spring Damped Angular Velocity
A_{2i-1}, A_{2i}	:	Motion Amplitude, $i = 1,2$
$\emptyset_{2i-1}, \emptyset_{2i}$	:	Phase Angle, $i = 1,2$
ζ_i	:	Damping Ratio, $i = 1,2$
T	:	Un-damped Bi-Linear Spring Free Vibration Time Period
T_d	:	Damped Bi-Linear Spring Free Vibration Time Period
T_{total}	:	Time of motion considered for 2 DoF.
δ	:	Damped Bi-Linear Spring Free Vibration Logarithmic Decrement
ω_i^k	:	Un-damped 2 DoF Free Vibration Angular Velocity for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
u_i^k	:	Displacement of 2 DoF for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
v_i^k	:	Velocity of 2 DoF for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
R_i^k	:	Amplitude of un-damped 2 DoF Free Vibration for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
ω_i^k	:	Un-damped 2 DoF Free Vibration Angular Velocity for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
$\emptyset_i^k$	:	Un-damped 2 DoF Free Vibration Phase Angle for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
r_i^k	:	Eigen Vector Ratio of 2 DoF for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
A^k, B^k, C^k, D^k	:	Solution Coefficient for Damped 2 DoF Free Vibration for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
α_i^k	:	Real Part of Eigen Value for Damped 2 DoF Free Vibration for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
β_i^k	:	Imaginary Part of Eigen Value for Damped 2 DoF Free Vibration for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$
γ_i^k	:	Real Part of Eigen Vector Ratio for Damped 2 DoF Free Vibration for k^{th} Configuration, $i = 1,2$ and $k = 1,2,3,4$

δ_i^k	:	Imaginary Part of Eigen Vector Ratio for Damped 2 DoF Free Vibration for k^{th} Configuration, $i = 1{,}2$ and $k = 1{,}2{,}3{,}4$

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